MATH 54 – MIDTERM 3 – SOLUTIONS

PEYAM RYAN TABRIZIAN

1. *(10 points, 2 points each)*

Label the following statements as **T** or **F**. **Write your answers in the box below!**

(a) **TRUE** If A is diagonalizable, then A^3 is diagonalizable.

($A = PDP^{-1}$, so $A^3 = PD^3P = \widetilde{P}\widetilde{D}\widetilde{P}^{-1}$, where $\widetilde{P} = P$ and $\widetilde{D} = D^3$, which is diagonal)

(b) **TRUE** If A is a 3×3 matrix with 3 (linearly independent) eigenvectors, then A is diagonalizable

(This is one of the facts we talked about in lecture, the point is that to figure out if A is diagonalizable, look at the eigenvectors)

(c) **TRUE** If A is a 3×3 matrix with eigenvalues $\lambda = 1, 2, 3$, then A is invertible

(No eigenvalue which is 0, so by the IMT, A is invertible)

(d) **TRUE** If A is a 3×3 matrix with eigenvalues $\lambda = 1, 2, 3$, then A is (always) diagonalizable

(this is the useful test we've been talking about in lecture, A is diagonalizable since it has 3 distinct eigenvalues)

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(e) **FALSE** If A is a 3×3 matrix with eigenvalues $\lambda = 1, 2, 2$, then A is (always) not diagonalizable

(Take $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, it is diagonal, hence diagonalizable)

2. *(15 points)* Label the following statements as **TRUE** or **FALSE**. In this question, you **HAVE** to justify your answer!!!

(a) **FALSE** If A is diagonalizable, then it is invertible.

For example, take $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. It is diagonalizable **because it is diagonal**, but it is not invertible!

(b) **FALSE** If A is invertible, then A is diagonalizable

Take $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (this is the 'magic counterexample' we talked about in lecture). It is invertible because $det(A) = 1 \neq 0$. To show it is not diagonalizable, let's find the eigenvalues and eigenvectors of A:

Eigenvalues:

$$det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 = 0$$

Which gives us $\lambda = 1$.

Eigenvectors:

$$Nul(I - A) = Nul \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

Which gives $-y = 0$, so $y = 0$, hence:

$$Nul(I - A) = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \right\} = Span \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

Since there is only one (linearly independent) eigenvector, A is not diagonalizable!

3. *(30 points)* Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$, where:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Eigenvalues:

$$\begin{aligned}
det(\lambda I - A) =& \begin{vmatrix} \lambda - 1 & -1 & -1 \\ 1 & \lambda - 3 & -1 \\ 1 & -1 & \lambda - 3 \end{vmatrix} \\
=&(\lambda - 1) \begin{vmatrix} \lambda - 3 & -1 \\ -1 & \lambda - 3 \end{vmatrix} - (-1) \begin{vmatrix} 1 & -1 \\ 1 & \lambda - 3 \end{vmatrix} + (-1) \begin{vmatrix} 1 & \lambda - 3 \\ 1 & -1 \end{vmatrix} \\
=&(\lambda - 1)((\lambda - 3)^2 - 1) + (\lambda - 3) + 1 - (-1 - (\lambda - 3)) \\
=&(\lambda - 1)(\lambda^2 - 6\lambda + 9 - 1) + \lambda - 3 + 2 + \lambda - 3 \\
=&\lambda^3 - 6\lambda^2 + 8\lambda - \lambda^2 + 6\lambda - 8 + 2\lambda - 4 \\
=&\lambda^3 - 7\lambda^2 + 16\lambda - 12
\end{aligned}$$

Now by the rational roots theorem, the only numbers a which divide -12 are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$, and the only numbers b which divide -1 are ± 1, hence by the rational roots theorem we should try $\lambda = \frac{a}{b} = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$.

After trying some roots, you should get $\boxed{\lambda = 2}$ works! Now use long division:

$$
\begin{array}{r|l}
 & X^2 \quad -5X \quad +6 \\
\hline
X-2 & X^3 - 7X^2 + 16X - 12 \\
 & -X^3 + 2X^2 \\
 & -5X^2 + 16X \\
 & 5X^2 - 10X \\
 & 6X - 12 \\
 & -6X + 12 \\
 & 0
\end{array}
$$

So $\lambda^3 - 7\lambda^2 + 16\lambda - 12 = (\lambda - 2)(\lambda^2 - 5\lambda + 6) = (\lambda - 2)(\lambda - 2)(\lambda - 3) = (\lambda - 2)^2(\lambda - 3)$

Hence the eigenvalues are $\boxed{\lambda = 2, 3}$

Eigenvectors:

$\lambda = 2$:

$$Nul(2I - A) = Nul \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} = Nul \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

But then $x - y - z = 0$, so $x = y + z$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y + z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Hence:

$$Nul(2I - A) = Span \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$\lambda = 3$:

$$Nul(3I - A) = Nul \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} = Nul \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix} = Nul \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

But then $x = z$ and $y = z$, so:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ z \\ z \end{bmatrix} = z \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Hence:

$$Nul(3I - A) = Span\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

<u>Answer:</u>

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

4. *(25 points)* Solve the following system $\mathbf{x}' = A\mathbf{x}$, where:

$$A = \begin{bmatrix} 0 & 5 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

<u>Eigenvalues:</u>

$$det(\lambda I - A)) = \begin{vmatrix} \lambda & -5 & 0 \\ 1 & \lambda - 4 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2)\left[\lambda(\lambda - 4) + 5\right] = (\lambda - 2)(\lambda^2 - 4\lambda + 5) = 0$$

Which gives $\lambda = 2$ and $\lambda^2 - 4\lambda + 5 = 0$, so $(\lambda - 2)^2 + 1 = 0$, so $\lambda = 2 \pm i$

Hence the eigenvalues are $\boxed{\lambda = 2, 2 \pm i}$

<u>Eigenvectors:</u>

<u>$\lambda = 2$</u>

$$Nul(2I - A) = Nul \begin{bmatrix} -2 & 5 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = Nul \begin{bmatrix} 1 & 2 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix} = Nul \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

But then $x = 0, y = 0$, and so:
And so:

$$Nul(2I - A) = \left\{ \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \right\} = Span \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

This tells us that $\mathbf{x}(t) = e^{2t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is a solution to the differential equation!

<u>$\lambda = 2 + i$</u>

$$Nul((2+i)I - A) = Nul \begin{bmatrix} 2+i & -5 & 0 \\ 1 & -2+i & 0 \\ 0 & 0 & i \end{bmatrix} = Nul \begin{bmatrix} 1 & \frac{-5}{2+i} & 0 \\ 1 & -2+i & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

However, $\frac{-5}{2+i} = \frac{-5(2-i)}{(2+i)(2-i)} = \frac{-5(2-i)}{4+1} = -2 + i$, so:

$$Nul((2+i)I - A) = Nul \begin{bmatrix} 1 & -2+i & 0 \\ 1 & -2+i & 0 \\ 0 & 0 & 1 \end{bmatrix} = Nul \begin{bmatrix} 1 & -2+i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

But then $x + (-2+i)y = 0$ and $z = 0$, so $x = (2-i)y$, and:

$$Nul((2+i)I - A) = \left\{ \begin{bmatrix} (2-i)y \\ y \\ 0 \end{bmatrix} \right\} = Span \left\{ \begin{bmatrix} 2-i \\ 1 \\ 0 \end{bmatrix} \right\}$$

This tells us that $\mathbf{x}(t) = e^{(2+i)t} \begin{bmatrix} 2-i \\ 1 \\ 0 \end{bmatrix}$ is a solution to the differential equation! But we can simplify this:

$$\begin{aligned} e^{(2+i)t} \begin{bmatrix} 2-i \\ 1 \\ 0 \end{bmatrix} &= \left(e^{2t}\cos(t) + ie^{2t}\sin(t)\right) \left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right) \\ &= \left(e^{2t}\cos(t) \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - e^{2t}\sin(t) \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right) + i \left(e^{2t}\sin(t) \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + e^{2t}\cos(t) \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right) \end{aligned}$$

And splitting into real and imaginary parts, and using the solution found for $\lambda = 2$, we get that:

<u>General Solution:</u>

$$\mathbf{x}(t) = Ae^{2t}\begin{bmatrix}0\\0\\1\end{bmatrix} + B\left(e^{2t}\cos(t)\begin{bmatrix}2\\1\\0\end{bmatrix} - e^{2t}\sin(t)\begin{bmatrix}-1\\0\\0\end{bmatrix}\right) + C\left(e^{2t}\sin(t)\begin{bmatrix}2\\1\\0\end{bmatrix} + e^{2t}\cos(t)\begin{bmatrix}-1\\0\\0\end{bmatrix}\right)$$

5. *(20 points, 10 points each)*
Find the general solution to $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$, where:

$$A = \begin{bmatrix}1 & 2\\0 & 3\end{bmatrix}, \mathbf{f}(t) = \begin{bmatrix}e^{4t}\\e^{4t}\end{bmatrix}$$

Note: You may use the fact that the general solution to $\mathbf{x}' = A\mathbf{x}$ is: $\mathbf{x}_0(t) = Ae^{t}\begin{bmatrix}1\\0\end{bmatrix} + Be^{3t}\begin{bmatrix}1\\1\end{bmatrix}$

(a) *(10 points)* Using undetermined coefficients

Particular solution: Usually you would guess y_p to be ae^{4t}, so here:

Guess $\mathbf{x}_p(t) = \mathbf{a}e^{4t} = \begin{bmatrix}Ae^{4t}\\Be^{4t}\end{bmatrix}$

Plug this into $\mathbf{x}_p' = A\mathbf{x}_p + \mathbf{f}$:

$$\begin{bmatrix}4Ae^{4t}\\4Be^{4t}\end{bmatrix} = \begin{bmatrix}1 & 2\\0 & 3\end{bmatrix}\begin{bmatrix}Ae^{4t}\\Be^{4t}\end{bmatrix} + \begin{bmatrix}e^{4t}\\e^{4t}\end{bmatrix}$$

which gives us:

$$\begin{bmatrix}4Ae^{4t}\\4Be^{4t}\end{bmatrix} = \begin{bmatrix}Ae^{4t} + 2Be^{4t} + e^{4t}\\3Be^{4t} + e^{4t}\end{bmatrix}$$

Which gives us: $4A = A + 2B + 1$ and $4B = 3B + 1$.

Which gives us $B = 1$, and $A = 1$

$$\mathbf{x}_p(t) = \begin{bmatrix}e^{4t}\\e^{4t}\end{bmatrix}$$

General solution: Using the **Note** at the beginning of the problem:

$$\mathbf{x}(t) = Ae^{t}\begin{bmatrix}1\\0\end{bmatrix} + Be^{3t}\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}e^{4t}\\e^{4t}\end{bmatrix}$$

(b) *(10 points)* Using variation of parameters

Particular solution:

Using the **Note**, we get that the Wronskian matrix is:

$$\widetilde{W}(t) = \begin{bmatrix}e^{t} & e^{3t}\\0 & e^{3t}\end{bmatrix}$$

Now suppose $\mathbf{x}_p(t) = v_1(t)\begin{bmatrix}e^{t}\\0\end{bmatrix} + v_2(t)\begin{bmatrix}e^{3t}\\e^{3t}\end{bmatrix}$, then use the variation of parameters formula:

$$\widetilde{W}(t)\begin{bmatrix}v_1'\\v_2'\end{bmatrix} = \mathbf{f}(t)$$

Here we get:

$$\begin{bmatrix}e^{t} & e^{3t}\\0 & e^{3t}\end{bmatrix}\begin{bmatrix}v_1'\\v_2'\end{bmatrix} = \begin{bmatrix}e^{4t}\\e^{4t}\end{bmatrix}$$

So:

$$\begin{bmatrix}v_1'\\v_2'\end{bmatrix} = \begin{bmatrix}e^{t} & e^{3t}\\0 & e^{3t}\end{bmatrix}^{-1}\begin{bmatrix}e^{4t}\\e^{4t}\end{bmatrix}$$

But $\begin{bmatrix}e^{t} & e^{3t}\\0 & e^{3t}\end{bmatrix}^{-1} = \frac{1}{e^{4t}}\begin{bmatrix}e^{3t} & -e^{3t}\\0 & e^{t}\end{bmatrix} = \begin{bmatrix}e^{-t} & -e^{-t}\\0e^{-3t} & \end{bmatrix}$ (here we used the formula $\begin{bmatrix}a & b\\c & d\end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix}d & -b\\-c & a\end{bmatrix}$)

Hence:

$$\begin{bmatrix}v_1'\\v_2'\end{bmatrix} = \begin{bmatrix}e^{-t} & -e^{-t}\\0 & e^{-3t}\end{bmatrix}\begin{bmatrix}e^{4t}\\e^{4t}\end{bmatrix} = \begin{bmatrix}0\\e^{t}\end{bmatrix}$$

Hence $v_1'(t) = 0$, so $v_1 = 0$, and $v_2'(t) = e^t$, so $v_2(t) = e^t$, and finally:

$$\begin{aligned}\mathbf{x}_p(t) =& v_1 \begin{bmatrix} e^t \\ 0 \end{bmatrix} + v_2(t) \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} \\ =& 0 \begin{bmatrix} e^t \\ 0 \end{bmatrix} + e^t \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} \\ =& \begin{bmatrix} e^{4t} \\ e^{4t} \end{bmatrix}\end{aligned}$$

General solution: Using the **Note** at the beginning of the problem:

$$\mathbf{x}(t) = Ae^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + Be^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} e^{4t} \\ e^{4t} \end{bmatrix}$$

Bonus *(2 points)*

(a) *(0 points, but it'll help you for (b))* What is the general solution of $y'' = -b^2 y$

$\boxed{y(t) = A_1 \cos(bt) + B \sin(bt)}$ (where A and B are constants)

(b) *(2 points)* Use (a) and the *ideas* we talked about in lecture about the matrix exponential function to solve the following system $\mathbf{x}'' = A\mathbf{x}$ (note the double prime), where:

$$A = \begin{bmatrix} 2 & -3 \\ 6 & 7 \end{bmatrix}$$

Hint: You may use the fact that $A = -B^2$, where $B = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$ as well as the fact that $B = PDP^{-1}$, where $P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

By analogy, you may suspect the general solution to be:

$$\mathbf{x}(t) = \cos(Bt)\mathbf{a} + \sin(Bt)\mathbf{b}$$

Where B is as above and $\mathbf{a} = \begin{bmatrix} A \\ B \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} C \\ D \end{bmatrix}$ How can we *define* $\cos(Bt)$ and $\sin(Bt)$? Well, you could use power series, but here it is easier just to use diagonalization, namely:

$$B = PDP^{-1} \Rightarrow \cos(Bt) = P\cos(Dt)P^{-1}, \quad \sin(Bt) = P\sin(Dt)P^{-1}$$

But here:

$$\begin{aligned}
\cos(Bt) &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \cos(t) & 0 \\ 0 & \cos(2t) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \cos(t) & 0 \\ 0 & \cos(2t) \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2\cos(t) - \cos(2t) & \cos(2t) - \cos(t) \\ 2\cos(t) - 2\cos(2t) & 2\cos(2t) - \cos(t) \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\sin(Bt) &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \sin(t) & 0 \\ 0 & \sin(2t) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \sin(t) & 0 \\ 0 & \sin(2t) \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2\sin(t) - \sin(2t) & \sin(2t) - \sin(t) \\ 2\sin(t) - 2\sin(2t) & 2\sin(2t) - \sin(t) \end{bmatrix}
\end{aligned}$$

<u>General solution:</u>

$$\begin{aligned}\mathbf{x}(t) =& \cos(Bt)\mathbf{a} + \sin(Bt)\mathbf{b} \\ =& A\begin{bmatrix} 2\cos(t) - \cos(2t) \\ 2\cos(t) - 2\cos(2t)\end{bmatrix} + B\begin{bmatrix} \cos(2t) - \cos(t) \\ 2\cos(2t) - \cos(t)\end{bmatrix} \\ &+ C\begin{bmatrix} 2\sin(t) - \sin(2t) \\ 2\sin(t) - 2\sin(2t)\end{bmatrix} + D\begin{bmatrix} \sin(2t) - \sin(t) \\ 2\sin(2t) - \sin(t)\end{bmatrix}\end{aligned}$$

which you can (but don't have to) simplify to:

$$\mathbf{x}(t) = A'\cos(t)\begin{bmatrix}1\\1\end{bmatrix} + B'\sin(t)\begin{bmatrix}1\\1\end{bmatrix} + C'\cos(2t)\begin{bmatrix}1\\2\end{bmatrix} + D'\sin(2t)\begin{bmatrix}1\\2\end{bmatrix}$$

<u>Other method:</u>

One of your fellow students, Jens Malmquist, found an **AMAZINGLY** slick solution to this problem, based on a proof I did in lecture. Here it is, and please savor it as much as I did :)

We want to solve $\mathbf{x}'' = A\mathbf{x}$. But since $A = -B^2$ and $B = PDP^{-1}$, we get $A = -PD^2P^{-1}$, hence we need to solve:

$$\begin{aligned}\mathbf{x}'' =& -PD^2P^{-1}\mathbf{x} \\ P^{-1}\mathbf{x}'' =& -D^2P^{-1}\mathbf{x} \\ \left(P^{-1}\mathbf{x}\right)'' =& -D^2\left(P^{-1}\mathbf{x}\right)\end{aligned}$$

Now let $\mathbf{y} = P^{-1}\mathbf{x}$. Then this becomes $\mathbf{y}'' = -D^2\mathbf{y}$.

But by (a), we get $\mathbf{y}(t) = \cos(Dt)\mathbf{a} + \sin(Dt)\mathbf{b}$. That is:

$$\mathbf{y}(t) = \begin{bmatrix}\cos(t) & 0 \\ 0 & \cos(2t)\end{bmatrix}\begin{bmatrix}A\\B\end{bmatrix} + \begin{bmatrix}\sin(t) & 0 \\ 0 & \sin(2t)\end{bmatrix}\begin{bmatrix}C\\D\end{bmatrix} = \begin{bmatrix}A\cos(t) + C\sin(t) \\ B\cos(2t) + D\sin(2t)\end{bmatrix}$$

But remember $\mathbf{y} = P^{-1}\mathbf{x}$, so $\mathbf{x} = P\mathbf{y}$, hence:

$$\mathbf{x}(t) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} A\cos(t) + C\sin(t) \\ B\cos(2t) + D\sin(2t) \end{bmatrix}$$
$$= \begin{bmatrix} A\cos(t) + C\sin(t) + B\cos(2t) + D\sin(2t) \\ A\cos(t) + C\sin(t) + 2B\cos(2t) + 2D\sin(2t) \end{bmatrix}$$
$$= A\cos(t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C\sin(t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + B\cos(2t)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + D\sin(2t)\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Rearranging (courtesy Jens: "or as Nicholas Cage would say: Put it in! The right file, according to alphabetical order! You know: A, B, C, D, $\cdots$! HUH!"), we get:

$$\mathbf{x}(t) = A\cos(t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + B\cos(2t)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + C\sin(t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + D\sin(2t)\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Congratulations again, Jens, this is amazing!!! :D